On the Spherical Hausdorff Measure in Step 2 Corank 2 sub-Riemannian Geometry

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Dedicated to A. Agrachev for his 60th birthday

Abstract

In this paper, we consider generic corank 2 sub-Riemannian structures, and we show that the Spherical Hausdorf measure is always a $\mathcal{C}^1$-smooth volume, which is in fact generically $\mathcal{C}^2$-smooth out of a stratified subset of codimension 7. In particular, for rank 4, it is generically $\mathcal{C}^2$. This is the continuation of a previous work by the auhors.

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1 Introduction

In this paper we consider sub-Riemannian structures $s = (\Delta, g)$ over an oriented n-dimensional manifold M. The distribution Δ has rank p and corank $k = n - p$, and g is a Riemannian metric over Δ. In most of the paper, $k = 2$. Moreover, the distribution is assumed to be 2-step bracket generating.

The set $\mathcal{S}$ of such (corank 2, 2-step bracket generating) sub-Riemannian structures over M is endowed with the $\mathcal{C}^\infty$ Whitney topology.

As it will be recalled in the next section, there is a natural smooth measure associated with the structure s, called the Popp measure (see [8]). It has been shown in [1] that the Radon-Nykodim derivative $f_{\mathcal{SP}}(\xi)$ of the spherical Haussdorf measure with respect to the Popp measure at a point ξ is just (universally) proportional to the inverse of the Popp-volume of the unit ball of the nilpotent approximation of s at ξ. Moreover, in the same paper, when $k = 1$, it is shown that $f_{\mathcal{SP}}(\xi)$ is a $\mathcal{C}^3$ function ($\mathcal{C}^4$ along curves), which is not $\mathcal{C}^5$ in general.

The nonsmoothness appears only in sub-Riemannian structures for which the nilpotent approximation depends on the point and can occur at points (called resonance points) where certain invariants of the structures coincide. The (high) degree 3 of differentiability is due to the fact that, in the corank 1 case, the conjugate locus of the nilpotent approximation coincides with the cut locus. This coincidence is no more true for higher corank. In particular, this is shown in [4], in the corank 2 case, and an explicit characterization of the cut-locus is given. In the same paper, as a simple byproduct, it has been shown that $f_{\mathcal{SP}}$ is generically $\mathcal{C}^1$ for $p = 4, k = 2$.

Starting from the explicit characterization of the cut-locus obtained in [4], in the current paper we go one step further and obtain the following result.

Theorem 1 *(step 2, corank 2) We have the following:*

1. the Radon-Nykodim derivative $f_{\mathcal{SP}}$ is always $\mathcal{C}^1$;

2. The Radon-Nykodim derivative $f_{\mathcal{SP}}$ is generically[1] *$\mathcal{C}^2$, out of a stratified set of codimension 7. In the particular case $p = 4$, there is an open-dense subset of $\mathcal{S}$ for which $f_{\mathcal{SP}}$ is $\mathcal{C}^2$-smooth.*

Remark 1 *In the case of a non-orientable manifold the Popp measure cannot be defined as a volume form, but just as a density. However, Theorem 1 still holds true since it is essentially local.*

Roughly speaking, $f_{\mathcal{SP}}$ depends on the maximum eigenvalue of a certain skew symmetric matrix (depending on the point) defining the nilpotent approximation of the structure at the point. (This eigenvalue is an invariant of the structure.) Hence, the study of the differentiability properties of $f_{\mathcal{SP}}$ requires a fine analysis of the regularity of the maximum eigenvalue of a family of skew symmetric matrices smoothly depending on parameters. When the maximum eigenvalue is simple at a point, then in a neighborhood of that point it is $\mathcal{C}^\infty$. A drop of regularity appears at points where the maximum eigenvalue is multiple. The $\mathcal{C}^1$ regularity can be obtained as a consequence of the fact that when the maximum eigenvalue is multiple, the cut locus coincides with the conjugate locus. [2] This fact does not permit to get the $\mathcal{C}^2$ result, which requires a deeper analysis. To treat double eigenvalues we need an adaptation of a deep result of Arnold [3], to the case of versal deformations of real skew-symmetric matrices. The case of triple eigenvalues is apparently extremely difficult and we do not treat it in this paper. However, the set of skew symmetric matrices with a triple eigenvalue is an algebraic subset of codimension 8 in skew symmetric matrices (we provide a proof of this technical fact in appendix). In the particular case of rank 4 this set is generically empty.

The paper is organized as follows: in Section 2, we recall the definition of the Popp measure and that of the nilpotent approximation of $s = (\Delta, g)$. We avoid to recall all standard definitions of sub-Riemannian geometry since these are already given in [4]. Then, we recall the main result of [4] which gives the cut time for geodesics issued from the origin. This is our key point. In Section 3, we give the proof of Theorem 1. This proof uses a certain number of technical tools that are collected in appendix. In Appendix A.1 we recall certain basic facts about quaternions, which here represents a very convenient tool. In A.2 we study versal deformations of real skew-symmetric matrices. In A.3 we discuss the codimension of the set of skew symmetric matrices having either a double or a triple eigenvalue. In A.4 we prove a result that (generically) allow us to make a crucial change of coordinates. In A.5, we recall how to get a useful formula for the volume of the nilpotent ball in the corank 2 case.

2 Prerequisites

2.1 Nilpotent approximation

[1] In this theorem, genericity means that the property is satisfied for a subset of sub-Riemannian metrics that CONTAINS an open-dense set. Indeed, in the transversality arguments, we can always avoid the closure of certain Whitney-stratified bad-sets, in place of avoiding the bad-sets themselves.

[2] Notice that in the corank 1 case, the cut locus coincides with the conjugate locus at every point, see [1].

We define the nilpotent approximation in the two-step baracket generating case only. The tensor mapping:

$$[.,.] : \Delta_\xi \times \Delta_\xi \to T_\xi M/\Delta_\xi, \tag{1}$$

is skew symmetric. Then, for any $Z^* \in (T_\xi M/\Delta_\xi)^*$ we have:

$$Z^*([X,Y]+\Delta_\xi) = < A_{Z^*}(X), Y >_g$$

for some g-skew-symmetric endomorphism A_{Z^*} of Δ_ξ. The mapping $Z^* \to A_{Z^*}$ is linear, and its image is denoted by $\mathcal{L}_\xi$.

The space $L_\xi = \Delta_\xi \oplus T_\xi M/\Delta_\xi$ is endowed with the structure of a 2-step nilpotent Lie algebra with the bracket:

$$[(V_1,W_1),(V_2,W_2)] = (0,[V_1,V_2]+\Delta_q).$$

The associated simply connected nilpotent Lie group is denoted by G_ξ, and the exponential mapping $E_{xp} : L_\xi \to G_\xi$ is one-to-one and onto. By translation, the metric g_ξ over Δ_ξ allows to define a left-invariant sub-Riemannian structure over G_ξ, called the nilpotent approximation of (Δ, g) at ξ.

Any k-dimensional vector subspace $\mathcal{V}_\xi$ of $T_\xi M$, transversal to Δ_ξ, allows to identify L_ξ and G_ξ to $T_\xi M \simeq \Delta_\xi \oplus T_\xi M/\Delta_\xi$. If we fix $\xi_0 \in M$,we can chose linear coordinates x in Δ_{ξ_0} such that the metric g_{ξ_0} is the standard Euclidean metric, and for any linear coordinate system y in $\mathcal{V}_{\xi_0}$, there are skew-symmetric matrices $L_1,...,L_k \in so(p,\mathbb{R})$ such that te mapping 1 writes:

$$[X,Y]+\Delta_\xi = \begin{pmatrix} X'L_1Y \\ . \\ . \\ X'L_kY \end{pmatrix},$$

where X' denotes the transpose of the vector X.

This construction works for any Δ, but Δ is 2-step bracket generating iff the endomorphisms of Δ_ξ, L_i, $i = 1,...,k$ (respectively the matices L_i if coordinates y in $\mathcal{V}_\xi$ are chosen) are linearly independant.

2.2 Popp Measure

In the 2-step bracket generating case, the linear coordinates y in $T_\xi M/\Delta_\xi$ can be chosen in such a way that the endomorphisms L_i, $i = 1,...,k$ are orthonormal with respect to the Hilbert-Schmidt scalar product $< L_i, L_j >= \frac{1}{p}\mathrm{Trace}_g(L_i'L_j)$. This choice defines a canonical euclidean structure over $T_\xi M/\Delta_\xi$ and a corresponding volume in this space. Then using the Euclidean structure over Δ_ξ, we get a canonical eucildean structure over $\Delta_\xi \oplus T_\xi M/\Delta_\xi$. The choice of the subspace $\mathcal{V}_\xi$ induces an euclidean structure on $T_\xi M$ that depends on the choice of $\mathcal{V}_\xi$, but the associated volume over $T_\xi M$ is independant of this choice.

Definition 1 *This volume form on M is called the Popp measure.*

By construction, the Popp measure is a smooth volume form.

Let us recall a main result from [1].

Theorem 2 *(equiregular, any step, any corank) The value $f_{\mathcal{SP}}(\xi)$ at $\xi \in M$ of the Radon-Nykodim derivative of the spherical Hausdorf measure with respect to the Popp measure is equal to $2^Q/\hat{\mu}(\hat{B}_\xi)$, where Q is the Hausdorff dimension of the sub-Riemannian structure as metric space and $\hat{\mu}(\hat{B}_\xi)$ is the Popp volume of the unit ball of the nilpotent approximation at ξ.*

2.3 Geodesics and Cut-locus

We restrict to the corank 2 case. Here, we consider geodesics of the nilpotent approximation of $s = (\Delta, g)$ in $T_{\xi_0}M \simeq \mathbb{R}^n$, issued from the origin. A transversal subspace $\mathcal{V}_{\xi_0}$ is chosen, together with the linear Hilbert-Schmidt-orthonormal coordinates y in $\mathcal{V}_{\xi_0}$, and euclidean coordinates x in Δ_{ξ_0}. The geodesics are projections on $\mathbb{R}^n$ of trajectories of the smooth Hamiltonian H on $T^*\mathbb{R}^n$:

$$H(p^x, p^y, x, y) = \sup_{u \in \mathbb{R}^p} (-\|u\|^2 + \sum_{i=1}^{p} p_i^x u_i + p_1^y x' L_1 u + p_2^y x' L_2 u). \tag{2}$$

where p^x, p^y are the coordinates dual to x, y. Geodesics are arclength-parametrized as soon as the initial covector $(p^x(0), p^y(0))$ verifies $H(p^x(0), p^y(0), x(0), y(0)) = 1/2$. For geodesics issued from the origin, this condition reads $\|u(0)\| = \|p^x(0)\| = 1$, where the norm $\| \cdot \|$ is the one induced by duality on $\Delta^*_{\xi_0}$.

Note that p_1^y, p_2^y are constant along geodesics, since the Hamiltonian (2) does not depend on the y-coordinates.

The following result is shown in [4], and is crucial for the proof of our result.

Theorem 3 *The cut time t_{cut} of the arclength-parametrized geodesic issued from the origin and corresponding to the initial covectors $(p^x(0), p^y)$ is given by:*

$$t_{cut} = \frac{2\pi}{\max(\sigma(p_1^y L_1 + p_2^y L_2)},$$

where $\max(\sigma(A))$ *denotes the maximum modulus of the eigenvalues of the skew symmetric matrix A. In general, the conjugate time is not equal to the cut time.*

Remark 2 *In fact the cut time is also conjugate if and only if the matrix $p_1^y L_1 + p_2^y L_2$ has a double maximum eigenvalue or $[L_1, L_2] = 0$.*

It turns out that the singularities of the Hausdorf measure appear due to collision between the two largest moduli of eigenvalues of the matrix $p_1^y L_1 + p_2^y L_2$. The set of skew-symmetric matrices that have a double eigenvalue is a codimension 3 algebraic subset of $so(p, \mathbb{R})$ (see Appendix A.3). Then, from the tranversality theorems ([2]), for generic (open, dense) sub-Riemannian structures, the set $\overline{\Sigma_2}$ of points of M such that $p_1^y L_1 + p_2^y L_2$ has a double (at least) eigenvalue for some $p_1^y, \underline{p_2^y}$ has codimension 2 in M. The problems of smoothness of the Hausdorf measure will occur on $\overline{\Sigma_2}$ only.

Along the paper we set, for the geodesic under consideration:

$$p_1^y = r\cos(\theta),\ p_2^y = r\sin(\theta),\ A_\xi(\theta, r) = \frac{2\pi}{\max(\sigma(p_1^y L_1 + p_2^y L_2))},\ \text{where } \xi = (x, y) \in M.$$

It is known ([6, 7, 9]) that $A_\xi(\theta, r)$ is a Lipschitz function of all parameters ξ, θ, r. We write also $A_\xi(\theta) = A_\xi(\theta, 1)$.

3 Proof of Theorem 1

For a fixed point $\xi_0 = (x_0, y_0) \in M$, let us consider the exponential mapping $\mathcal{E}$ associated with the nilpotent approximation at ξ_0, where x, y are coordinates as in Section 2.2:

$$\mathcal{E}_t(p_0^x, p_0^y) = \pi(e^{t\vec{H}}(p_0^x, p_0^y, \xi_0)),$$

where $\pi : T^*M \to M$ is the canonical projection, and $\vec{H}$ is the Hamiltonian vector field associated with the Hamiltonian (2). Here (p_0^x, p_0^y) are initial covectors satisfying $H(p_0^x, p_0^y, \xi_0) = 1/2$.

As above we have $p^y(t) = p_0^y = (p_1^y, p_2^y) = (r\cos(\theta), r\sin(\theta))$. Also, by homogeneity, $\mathcal{E}_t(p_0^x, p_0^y) = \mathcal{E}_1(t\ p_0^x, t\ p_0^y)$.

In our paper [4], the following formula is given for the volume V_ξ at a point $\xi \in M$ of the unit ball of the nilpotent approximation. For the benefit of the reader, this formula is established here in Appendix A.5.

$$V_\xi = \int_0^{2\pi} \int_0^{A_\xi(\theta)} \int_B J(p_0^x, \theta, r, \xi) dp_0^x\ dr\ d\theta \tag{3}$$

where B is the unit ball in the euclidean p-dimensional p_0^x-space, and $J(p_0^x, \theta, r, \xi)$ is the jacobian determinant of $\mathcal{E}_1(p_0^x, r\cos(\theta), r\sin(\theta))$.

We set $f_\xi(\theta, r) = \int_B J(p_0^x, \theta, r, \xi) dp_0^x$, and $W_\xi(\theta) = \int_0^{A_\xi(\theta)} f_\xi(\theta, r) dr$. If we show that $W_\xi(\theta)$ is $\mathcal{C}^1$ or $\mathcal{C}^2$ w.r.t (θ, ξ), it will imply that V_ξ is $\mathcal{C}^1$ or $\mathcal{C}^2$ w.r.t ξ.

In a neighborhood of a fixed $(\theta_0, \xi_0) \in S^1 \times M$ we have,

$$\begin{aligned} W_\xi(\theta) &= \int_0^{A_\xi(\theta)} f_\xi(\theta, r) dr \qquad (4) \\ &= \int_0^{A_{\xi_0}(\theta_0)} f_\xi(\theta, r) dr \ + \int_{A_{\xi_0}(\theta_0)}^{A_\xi(\theta)} f_\xi(\theta, r) dr \\ &=: (I) + (II). \end{aligned}$$

The term (I) is smooth. We are then left to study the smoothness of $II(\xi, \theta)$.

3.1 Proof of the fact that $W_\xi(\theta)$ is always $\mathcal{C}^1$

Setting $z = (\theta, \xi)$, $z_0 = (\theta_0, \xi_0)$, and $f(z, r) = f_\xi(\theta, r)$, $A(z) = A_\xi(\theta)$, the tangent mapping to $II(\xi, \theta)$, at (θ_0, ξ_0) is

$$D\ II(z_0)(h) = \sum_{i=1}^{n+1} f(z_0, A(z_0)) \frac{\partial A}{\partial z_i}(z_0) h_i. \tag{5}$$

This last expression makes sense, and is continuous w.r.t z_0 for the following reasons: first as we said, $A(z)$ is Lipschitz-continuous, then the derivatives are bounded. Moreover at points z_0 such that A is not differentiable, $f(z_0, A(z_0))$ vanishes. This last point follows from the fact that when the eigenvalue of $A(z_0)$ having maximum modulus is multiple then the conjugate time is equal to the cut time, which makes the jacobian determinant $J(p_0^x, \theta_0, A(\theta_0, \xi_0), \xi_0)$ vanish for all p_0^x. This comes from the section II.3 1 in the paper [1].

Remark 3 *In fact, it follows from the same paper that, if $A(z_0)$ corresponds to a multiple eigenvalue, then the rank of $J_{\xi_0}(p_0^x, \theta_0, A(\theta_0, \xi_0)) = J(p_0^x, \theta_0, A(\theta_0, \xi_0), \xi_0)$ drops by 2 at least, independently of p_0^x. This point will be very important in the next section.*

This ends the proof.

3.2 Proof of the $\mathcal{C}^2$ result

It follows from the transversality theorems ([2, 5]) and from Lemma 2 and Lemma 3 in the Appendix, that there exists an open dense subset of sub-Riemannian metrics, still denoted by $\mathcal{S}$, such that all elements s of $\mathcal{S}$ meet: the set $U_s \subset S^1 \times M$ of (θ, ξ) such that $A_\xi(\theta)$ corresponds to a triple (at least) eigenvalue is a locally finite union of manifolds, regularly embedded, of codimension 8 in $S^1 \times M$, and the set $\tilde{U}_s \subset S^1 \times M$ of (θ, ξ) such that $A(\theta, \xi)$ corresponds to a double (and not triple) eigenvalue is a locally finite union of manifolds, of codimension 3.

We want to show the following property (P), for a (smaller) generic (residual in the Whitney topology) set $\mathcal{S}_0$ of sub-Riemannian metrics over M:

(P) the partial derivatives $D_i(z) = f(z, A(z))\frac{\partial A}{\partial z_i}(z)$ from (5) are $\mathcal{C}^1$ in a neighborhood of all points z_0 such that $A(z_0)$ corresponds to a double (and not triple) eigenvalue.

To do this, we fix s_0 and $z_0 \in \tilde{U}_{s_0}$ and we consider a (mini)versal deformation of $L(\xi_0, \theta_0) = L_1(\xi_0)\cos(\theta_0) + L_2(\xi_0)\sin(\theta_0) = L(z_0)$, as introduced in Appendix A.2. It follows that:

$$L(\xi, \theta) = L(z) = g(z)^{-1}\mathcal{T}(z)g(z)$$

where $g(z)$ belongs to the orthogonal group and[3] the functions $g(\cdot), \lambda(\cdot), q(\cdot), \Delta(\cdot)$ are smooth with respect to z.

The following crucial Lemma is proved in Appendix A.4

Lemma 1 *The property*

(R): the map $S^1 \times M \ni z \mapsto q(z) \in \mathbb{R}^3$, has rank 3 at every $z \in \tilde{U}_s$,

is residual in $\mathcal{S}$.

Let us call $\mathcal{S}_0$ the subset of $\mathcal{S}$ for which (R) holds. If s_0 is fixed in $\mathcal{S}_0$ and $z_0 \in \tilde{U}_{s_0}$ then, locally around z_0, we can find a system of coordinates in $S^1 \times M$ in such a way that the three first coordinates, z_1, z_2, z_3 become the three components of $q(z)$. Note that these 3 coordinates vanish at z_0.

Locally, the codimension 3 manifold $\tilde{U}_{s_0}$ is determined by the equations $z_1 = z_2 = z_3 = 0$.

As we said in Remark 3, the rank of $J(p_0^x, z, A(z))$ drops by 2 at least, independantly of p_0^x, at each point $z \in \tilde{U}_{s_0}$. Formula (7) in the appendix tell us that $A(z) = \frac{2\pi}{\lambda(z)+\sqrt{z_1^2+z_2^2+z_3^2}}$ where $\lambda(z)$ is smooth and nonzero. We set $\hat{z}_4 = (z_4, ..., z_{n+1})$ and $\hat{z}_1 = (z_1, z_2, z_3)$.

The Jacobian determinant $J(p_0^x, z, r)$ can be written as

$$V_1(p_0^x, z, r) \wedge \ldots \wedge V_{n+1}(p_0^x, z, r),$$

for certain smooth $n+1$-dimensional vectors $V_1(p_0^x, z, r) \ldots V_{n+1}(p_0^x, z, r)$.

For all p_0^x, at points (z, r) such that $\hat{z}_1 = 0$, $r = A(z) = \frac{2\pi}{\lambda(z)}$, the vectors $V_1 \ldots V_{n+1}$ have rank $n-1$ at most. Then

[3] $\mathcal{T}(z)$ plays the role of $T(\mu(z))$ in Appendix A.2. $\mathcal{T}(z)$ is the block-diagonal matrix $\mathcal{T}(z) = \mathrm{Bd}(\lambda(z)\hat{q} + q(z), \Delta(z))$. Here, following the notation introduced in the appendix, q is a pure quaternion, $\hat{q}$ is a pure skew-quaternion, $\Delta(z)$ is a 2×2 block diagonal skew-symmetric matrix and $\lambda(z)$ is a nonzero real number.

$$0=\frac{\partial J}{\partial z_i}=\frac{\partial V_1}{\partial z_i}\wedge V_2\wedge\ldots\wedge V_{n+1}+V_1\wedge\frac{\partial V_2}{\partial z_1}\wedge\ldots\wedge V_{n+1}+\ldots$$

and

$$0=\frac{\partial J}{\partial r}=\frac{\partial V_1}{\partial r}\wedge V_2\wedge\ldots\wedge V_{n+1}+V_1\wedge\frac{\partial V_2}{\partial r}\wedge\ldots\wedge V_{n+1}+\ldots$$

It follows that J, $\frac{\partial J}{\partial z_i}$, $\frac{\partial J}{\partial r}$ vanish at all (z,r) with $\hat{z}_1=0$, $r=A(z)=\frac{2\pi}{\lambda(z)}$.

Therefore $f(z,r)=\int_{B^1}J_\xi(p_0^x,\theta,r)\,dp_0^x$ is a quadratic expression in the variable $\hat{z}_1$, $r-\frac{2\pi}{\lambda(z)}$ depending smoothly on z, r:

$$f(z,r)=\tilde{Q}_{z,r}(\hat{z}_1,r-\frac{2\pi}{\lambda(z)}). \tag{6}$$

Now we study the continuity of the second partial derivatives of $W_\xi(\theta)=\int_0^{A(\theta_0,\xi_0)}f_\xi(\theta,r)dr$ $+\int_{A(\theta_0,\xi_0)}^{A(\theta,\xi)}f_\xi(\theta,r)dr$, or with the new notations, $W(z)=\int_0^{A(z_0)}f(z,r)dr+\int_{A(z_0)}^{A(z)}f(z,r)dr$.

The first partial derivatives, at any point z_0 were:

$$\begin{aligned}\frac{\partial W}{\partial z_i}(z_0)&=\int_0^{A(z_0)}\frac{\partial}{\partial z_i}f(z_0,r)dr+f(z_0,A(z_0))\frac{\partial A}{\partial z_i}(z_0),\\&=:III(z_0)+IV(z_0)\end{aligned}$$

To show that $\frac{\partial III(z)}{\partial zj}$ exists and is continuous, we proceed exactly as in Section 3.1, using the fact that $\frac{\partial}{\partial z_j}f(z,r)$ also vanishes at ($\hat{z}_1=0$, $r=\frac{2\pi}{\lambda(z)}$).

The more difficult point is to show that $\frac{\partial IV(z)}{\partial zj}$ exists and is continuous.

$$\frac{\partial IV(z)}{\partial z_j}=\frac{\partial}{\partial z_j}\left(f(z,A(z))\frac{\partial A(z)}{\partial z_i}\right).$$

We get:

$$\begin{aligned}\frac{\partial IV}{\partial z_j}(z)&=\frac{\partial f}{\partial z_j}(z,A(z))\frac{\partial A(z)}{\partial z_i})+\frac{\partial f}{\partial r}(z,A(z))\frac{\partial A(z)}{\partial z_i}\frac{\partial A(z)}{\partial z_j}+f(z,A(z))\frac{\partial^2A(z)}{\partial z_i\partial z_j}.\\&=:V(z)+VI(z)+VII(z).\end{aligned}$$

The cases of $V(z),VI(z)$ are obvious, since again $\frac{\partial A(z)}{\partial z_i}$ is bounded, and the functions $\frac{\partial f}{\partial z_j}(z,A(z))$, $\frac{\partial f}{\partial r}(z,A(z))$ are continuous and go to zero when $\hat{z}_1$ tends to zero. The only difficulty is the case of $VII(z)$.

Remind that $A(z)=\frac{2\pi}{\lambda(z)+||\hat{z}_1||.}$ where $\lambda(z)$ is nonzero, smooth. Then the only problem may occur for $i=1,2,3$.

Let us consider only the 2 cases: (1) $i=1,j=4$, (2) $i=1,j=2$, the other being similar.

Case (1): $\frac{\partial A(z)}{\partial z_1}=\frac{-2\pi}{(\lambda(z)+\|\hat{z}_1\|)^2}(\frac{\partial\lambda}{\partial z_1}+\frac{z_1}{\|\hat{z}_1\|})$, and $\frac{\partial^2A(z)}{\partial z_1\partial z_4}$ is bounded. It is multiplied by $f(z,A(z))$, which tends to zero when $\hat{z}_1$ tends to zero. Then it is zero at points $\hat{z}_1=0$, and it is continuous.

Case(2): $\frac{\partial A(z)}{\partial z_1}=\frac{-2\pi}{(\lambda(z)+\|\hat{z}_1\|)^2}(\frac{\partial\lambda}{\partial z_1}+\frac{z_1}{\|\hat{z}_1\|})$, and $\frac{\partial^2A(z)}{\partial z_1\partial z_2}=C(z)+D(z)\frac{z_1z_2}{||\hat{z}_1||^3}$, where $C(z)$ is bounded, $D(z)$ is continuous. Then, the question is the continuity to zero of $\varphi(z)=\frac{f(z,A(z))}{||\hat{z}_1||}$, in a neighborhood of the set $E=\{\hat{z}_1=0\}$. Let us use Formula (6). It gives $f(z,A(z))=\tilde{Q}_{z,r}(\hat{z}_1,A(z)-\frac{2\pi}{\lambda(z)})$. But $A(z)=\frac{2\pi}{\lambda(z)+||\hat{z}_1||}$, then, $A(z)-\frac{2\pi}{\lambda(z)}=\psi(z)||\hat{z}_1||$, where $\psi(z)$ is continuous. It follows that $\varphi(z)$ tends to zero when $\hat{z}_1$ tends to zero. The sub-Riemannian volume is $\mathcal{C}^2$ in a neighborhood of $\tilde{U}_{s_0}$.

It follows that $f_{\mathcal{SP}}(\xi)$ is generically $\mathcal{C}^2$ except on a bad set of codimension 8 in $S^1 \times M$, and the theorem is proved. In the case $n = 6$, the bad set is generically empty in $S^1 \times M$ and property (R) is open dense in $\mathcal{S}$.

A Appendix

A.1 Pure Quaternions in $so(4)$

In $so(4)$, it is natural and useful for computations to use quaternionic notations. Set:

$$i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\imath} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \hat{\jmath} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \hat{k} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The matrices i, j, k (resp. $\hat{\imath}, \hat{\jmath}, \hat{k}$) generate the so-called pure quaternions (resp. pure skew-quaternions), the space of which is denoted by Q (resp. $\hat{Q}$). The Lie algebra $so(4) = Q \oplus \hat{Q}$, and quaternions commute with skew-quaternions: $[Q, \hat{Q}] = 0$.

We endow $so(4)$ with the Hilbert-Schmidt scalar product: $< L_1, L_2 >= trace(L_1' L_2)$.

Then, $i, j, k, \hat{\imath}, \hat{\jmath}, \hat{k}$ form an orthonormal basis. The eigenvalues ω_1, ω_2 of $A = q + \hat{q}$ meet:

$$-(\omega_{1,2})^2 = (||q|| \pm ||\hat{q}||)^2. \tag{7}$$

As a consequence, an element $A \in so(4)$ has a double eigenvalue iff $A \in Q \cup \hat{Q}$.

A.2 Versal deformation of skew-symmetric matrices

The results of Arnold in [3] can be easily extended to the real smooth case ($\mathcal{C}^\infty$), for skew-symmetric matrices, under the action of the orthogonal group:

Theorem 4 *[3] Let $N(p)$ be a family of $n \times n$ matrices smoothly depending on p at $(\mathbb{R}^l, 0)$. Let O_N be the orbit of $N = N(0)$ under the action of $Gl(n, \mathbb{R})$ by conjugation. Let $T(\mu)$ be a smooth family of matrices, depending on the parameter $\mu \in \mathbb{R}^k$, such that the mapping $\mu \to T(\mu)$* ***transversally*** *intersects O_N at some $\tilde{N} = g^{-1} N g$. Then, there is a family of (smoothly depending on p) matrices $g(p)$ and a smooth mapping $p \to \mu(p)$, such that $N(p) = g(p)^{-1} T(\mu(p)) g(p)$. Moreover, for the transversal $T(\mu)$, one can choose the centralizer of N in $gl(n, \mathbb{R})$.*

We rephrase the result in the case of a skew-symmetric matrix N that has a double (but not triple) eigenvalue. Then, by section A.1, we can assume that N is (conjugate to) a block-diagonal $Bd(\alpha \hat{q}, \delta)$, where $\hat{q}$ is a unit skew-quaternion and δ is a block-diagonal skew symmetric matrix with 2×2 blocks and non multiple eigenvalues. The centralizer of $\hat{q}$ in $so(4, \mathbb{R})$ is the vector space of matrices of the form $\lambda \hat{q} + q$, where q varies over pure quaternions. Then, the centralizer of N in $so(n,\mathbb{R})$ is the space of block diagonal matrices $\mathrm{Bd}(\lambda\hat{q}+q,\Delta)$, where q varies over pure quaternions and Δ varies over 2×2 skew-symmetric block diagonal matrices.

Hence, we can find a smooth $g(p)\in SO(n,\mathbb{R})$, and a smooth $\mu(p)$ such that:

$$N(p) = g(p)^{-1}T(\mu(p))g(p), \quad \text{with} \qquad (8)$$
$$T(\mu) = \mathrm{Bd}(\lambda(\mu)\hat{q}+q(\mu),\Delta(\mu)).$$

The versal deformation $T(\mu)$ is not universal (which means that $\mu(p)$ is not uniquely determined by $N(p)$), however, the nondiagonal eigenvalues of $T(\mu)$ are given by the Formula (7). It follows that q is determined modulo conjugation by a unit quaternion. On the other hand, the functions $\lambda(\mu),\Delta(\mu)$ are smooth and $\lambda(\mu)$ is nonzero.

A.3 Codimension of double and triple eigenvalues

Lemma 2 *We have the following:*

(i) the set of skew symmetric matrices with a double eigenvalue is an algebraic subset of codimension 3 in skew symmetric matrices;

(ii) the set of skew symmetric matrices with a triple eigenvalue is an algebraic subset of codimension 8 in skew symmetric matrices.

The proof of (i) is given in the appendix of [10]. The proof of (ii) given hereafter is a generalization. We restrict ourself to the even dimensional case $so(2n)$, the odd dimensional case being similar.

We consider the set $\mathcal{D}$ of block-diagonal matrices D of the form

$$D = \mathrm{Bd}(\alpha J,\alpha J,\alpha J,\alpha_4 J,\dots,\alpha_n J),$$

of dimension $N=2n$, with $J=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and we show only that the union of the orbits under orthogonal conjugation of the elements of $\mathcal{D}$ has codimension 8 at least. To do this we consider generic elements of $\mathcal{D}$ only: for non-generic elements the dimension of the orbit is smaller.

To compute the dimension of the orbit $\mathcal{O}_D$ of D it is enough to compute the dimension of the stabilizer G of D, and then to compute the dimension of the Lie algebra $\mathcal{L}=\mathrm{Lie}(G)$, which is just the centralizer $\mathcal{C}$ of D.

By a direct computation one gets (if D is a generic element) that elements C of $\mathcal{C}$ are of the form

$$C = \mathrm{Bd}(A_1,\Delta),$$

where A_1 is 6×6 and Δ is block diagonal with 2×2 blocks. Both A_1 and Δ are skew-symmetric and A_1 is of the form

$$\begin{pmatrix} \alpha_1 J & B_{1,2} & B_{1,3} \\ -B_{1,2} & \alpha_2 J & B_{2,3} \\ -B_{1,3} & -B_{2,3} & \alpha_3 J \end{pmatrix},$$

and

$$B_{i,j} = \begin{pmatrix} \beta_{i,j} & \gamma_{i,j} \\ -\gamma_{i,j} & \beta_{i,j} \end{pmatrix}$$

Then, $\dim(C)=\dim(G)=n-3+9=n+6$. Therefore $\dim(\mathcal{O}_D)=m-n-6$, where $m=n(2n-1)$ is the dimension of $so(2n)$. The dimension of $\mathcal{D}$ is $n-2$. Hence the dimension of the union of the orbits through points of $\mathcal{D}$ is $m-n-6+n-2=m-8$.

A.4 Proof of Lemma 1: Genericity of (R)

We consider the set $\mathcal{S}$ of corank-2 sub-Riemannian metrics on a fixed manifold M, equipped with the Whitney topology. The result being essentially local, we may assume that M is an open set of $\mathbb{R}^n$, with global coordinates ξ, and that our sub-Riemannian metrics are globally specified by an orthonormal frame, i.e. $s = (F_1, ..., F_p)$.

For the moment, we fix $s \in \mathcal{S}$. We consider two independant one forms ω_1, ω_2 on M, that vanish on Δ, and we set $\tilde{L}_i = d\omega_{i|\Delta}$, and L_i is the skew-symmetric matrix defined by $\tilde{L}_i$ via the metric, and moreover we impose (as in 2.2) that $L_1(\xi), L_2(\xi)$ are Hilbert-Schmidt-orthonormal. The matrices L_1, L_2 are defined uniquely modulo a rotation $\hat{L}_1 = \cos(\alpha(\xi))L_1 + \sin(\alpha(\xi))L_2$, $\hat{L}_2 = -\sin(\alpha(\xi))L_1 + \cos(\alpha(\xi))L_2$. They are the same as the matrices L_i in Section 2.1 and they meet: $(L_i)_{k,l} = \omega_i([F_k, F_l]) = d\omega_i(F_k, F_l)$.

In coordinates, we set $z = (\xi, \theta)$, and $A(z) = \cos(\theta)L_1(\xi) + \sin(\theta)L_2(\xi)$.

We fix a point a point $z_0 = (\theta_0, \xi_0) \in \tilde{U}_s \subset S^1 \times M$, and we work in a neighborhood of z_0. By what has just beeen said, we can perform a constant rotation to have $\theta_0 = 0$.

Local coordinates $\xi = (x, y)$ in M around ξ_0 can be found, with $x_0 = 0$, $y_0 = 0$, such that:

1. $F_i(\xi_0) = \frac{\partial}{\partial x_i}$, $i = 1, \dots, p$
2. $\omega_j(\xi_0) = dy_j - x' L_j(\xi_0)dx$, $j = 1, 2$
3. $A(z_0) = L_1(\xi_0)$ is 2×2 block diagonal with decreasingly ordered (moduli of) eigenvalues. (For this last point, we use an (irrelevant) rotation in the distribution Δ_{ξ_0}, i.e. a constant rotation of the orthonormal frame)

Remark 4 *Note that at a point $z_0 = (\xi_0, \theta_0) \in \tilde{U}_s$, the two (moduli of) highest eigenvalues of $A(z_0)$ are equal. However, the whole construction here holds at each point of $S^1 \times M$.*

In these coordinates, we can write (locally) s in the following form: $F_i(\xi) = (e_i + B^i\xi)\frac{\partial}{\partial x} + x' L_1(\xi_0)e_i\frac{\partial}{\partial y_1} + x' L_2(\xi_0)e_i\frac{\partial}{\partial y_2} + O^2(\xi)$, where $e_i = (0, .., 1, .., 0)$ is the i^{th} coordinate vector in $\mathbb{R}^p$, where B^i is a $p \times n$ matrix, and $O^2(\xi)$ is a term of order 2 in ξ, i.e. $O^2(\xi)$ is in $\mathcal{I}^2$, where $\mathcal{I}$ is the ideal of smooth germs at 0 in $\mathbb{R}^n$, generated by the components ξ_i.

This choice of notations for the vector fields F_i is adapted to the transversality arguments we want to apply later. Note that $L_1(\xi_0)$, B^i are, in coordinates, components of the one-jet $j^1s(\xi_0)$ of s at ξ_0.

Define the $p \times p$ matrix U^r by $U^r_{i,j} = B^j_{i,r}$, and by Ω_1, Ω_2 the skew-symmetric matrices associated with the 2-forms $d\omega_{1|\Delta}(\xi_0), d\omega_{2|\Delta}(\xi_0)$ in the chosen coordinates. It is not hard to compute the tangent mappings $TL_1(\xi_0)$ and $TL_2(\xi_0)$ (we temporarily write $TL(\xi_0)$ and Ω for convenience):

$$TL(\xi_0)(e_r) = U^{r\prime}\Omega - \Omega' U^r, \quad r = 1, ..., n. \tag{9}$$

For this, one just uses $d \circ d = 0$, and $TL_{k,l}(\xi_0)(e_r) = d\omega(TF_k(e_r), F_l) + d\omega(F_k, TF_l(e_r))$, where $d\omega$ stands for $d\omega_1$ or $d\omega_2$.

On the other hand, we have, using the versality theorem in a neighborhood of z_0:

$$A(z) = \cos(\theta)L_1(\xi) + \sin(\theta)L_2(\xi) = H(z)\mathrm{Bd}(\lambda(z)\hat{q} + q(z), \Delta(z))H'(z), \tag{10}$$

in which we already assumed that $\theta_0 = 0$, and the coordinates $\xi = (x, y)$ were already chosen for $A(z_0)$ to be diagonal. Also, $H(z_0) = Id$.

Remark 5 *1. The decomposition (10) is not unique: the quaternion $q(z)$ is defined modulo conjugation by a unit quaternion, $\tilde{q}(z) = q_1(z)q(z)q_1(z)^{-1}$. However, the tangent mapping $Tq(z)$ is changed for $T\tilde{q}(z) = [Tq_1(z), q(z)] + q_1(z)Tq(z)q_1(z)^{-1}$. But on $\tilde{U}_s$, $q(0) = 0$, hence the rank of $Tq(z_0)$ remains unchanged.*

2. The decomposition can easily be made unique, by making (following Arnold [3]) some particular choice of a (mini)transversal to the centraliser of $A(z_0)$. For instance, one could chose the (Hilbert-Schmidt) orthogonal supplement to the centralizer of $A(z_0)$ through $A(z_0)$.

Let $\Pi_Q : so(n) \to Q \simeq \mathbb{R}^3$, be the projection associating to the matrices, the quaternionic components of the first 4×4 block on the diagonal.

By (10), we have:

$$Tq(z_0)(V) = \Pi_Q TA(z_0)(V) + \Pi_Q[TH'(z_0)(V), \mathrm{Bd}(z_0)]. \quad (11)$$

We can consider the fiber mapping $\pi_Q : J^1\mathcal{S}{\times}S^1 \to Q \times \mathcal{M}(3, n+1)$, $\pi_{Q,z_0} : (j^1 s(\xi_0), \theta_0) \to (q(z_0), Tq(z_0))$ ($\mathcal{M}(3, n+1)$ being the set of $3 \times (n+1)$ real matrices),

$$\pi_{Q,z_0}(L_1, L_2, B^i, i = 1, ..., r) = \{\Pi_Q(A(z_0)),\ \Pi_Q \circ TA(z_0)\}.$$

The following lemma is an easy consequence of (9), (even easier to prove if one considers that[4] $\Omega = \Omega_1 = L_1$, is 2×2 block diagonal, the 2 first blocks being both nonzero):

Lemma 3 *The mapping π_{Q,z_0} is a linear submersion.*

It follows from Lemma 3 that the mapping $\rho : J^1\mathcal{S}{\times}S^1 \to \mathbb{R}^3 \times \mathcal{M}(3, n+1)$, $(z_0, L_1, L_2, B^i, i = 1, ..., p) \to Tq(z_0)$ is a submersion.

The codimension d_0 of the algebraic set of $3 \times (n+1)$ matrices that have corank 1 at least is $d_0 = (n-1)$ [product of coranks in the $3 \times (n+1)$ matrices]. By Lemma 2, the set of skew-symmetric matrices that have double maximum eigenvalue is $d_1 = 3$. Therefore, by the transversality theorems [2], there is a residual subset of the set of sub-Riemannian metrics, for which the codimension of the set of $z = (\theta, \xi)$ in $S^1 \times M$ where $A(z)$ corresponds to a double eigenvalue, an property (R) holds at (θ, ξ), is a stratified set of codimension $d_0 + d_1 = n + 2$.

A.5 Volume of the unit ball

We keep the notations of Section 3.

$$\mathcal{E}_t(p_0^x, p_0^y) = \mathcal{E}_t(p_0^x, r, \theta) = \mathcal{E}_1(t\, p_0^x, t\, r, \theta),$$

ant $t_{cut} = A_\xi(\theta)/r$. The domain of $\mathcal{E}_t$ for the unit ball is

$$D_{\mathcal{E}_t} = \left\{(p_0^x, r, \theta, t) \mid \theta \in [0, 2\pi],\ p_0^x \in S^{p-1}, t \in \min(1, t_{cut}),\ r \in [0, +\infty[\right\},$$

where S^{p-1} denotes the unit Euclidean sphere in $\mathbb{R}^p$ and later B denotes the unit Euclidean ball $\mathbb{R}^p$. In this formula, the boundary of this set in the variables r, t is parametrized by r. Equivalently if we parametrize this boundary by t we get,

$$D_{\mathcal{E}_t} = \left\{(p_0^x, r, \theta, t) \mid \theta \in [0, 2\pi],\ p_0^x \in S^{p-1}, t \in [0, 1],\ r = A_\xi(\theta)/t\right\}.$$

Now set $\tilde{p} = t\, p_0^x$, $\tilde{r} = t\, r$. This implies for the domain $D_{\mathcal{E}_1}$ of $\mathcal{E}_1(\tilde{p}, \tilde{r}, \theta)$,

[4]Note that in fact, in the chosen coordinates, $\Omega_i = L_i$ since $\mathrm{F}_i(z_0) = e_i$ on Δ_{z_0}

$$D_{\mathcal{E}_1} = \{(\tilde{p}, \tilde{r}, \theta) \mid \theta \in [0, 2\pi],\ \tilde{p} \in B,\ \tilde{r} \in [0, A_\xi(\theta)]\}\,.$$

The volume of the unit ball of the nilpotent approximation at ξ is

$$V_\xi = \int_{\mathcal{E}_1(D_{\mathcal{E}_1})} \mathrm{Popp} = \int_{\mathcal{E}_1(D_{\mathcal{E}_1})} dx \wedge dy,$$

that is

$$V_\xi = \int_0^{2\pi} \int_0^{A_\xi(\theta)} \int_B J_{\mathcal{E}_1(\tilde{p}.\tilde{r},\theta) d\tilde{p}\, d\tilde{r}\, d\theta},$$

where $J_{\mathcal{E}_1}$ is the Jacobian determinant of $\mathcal{E}_1(\tilde{p}, \tilde{r}, \theta)$.

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References

[1] A. Agrachev, D. Barilari, U. Boscain, On the Hausdorff volume in sub-Riemannian geometry. Calculus of Variations and PDE's Volume 43, Numbers 3-4 March 2012.

[2] R. Abraham, R.J. Robbin, Transversal mappings and flows, W.A. Benjamin Inc., New-York-Amsterdam, 1967.

[3] V.I. Arnold, On Matrices depnding on parameters, Russian Math. Surveys, 26, 1971,2, 101-114.

[4] D. Barilari, U. Boscain, J.P. Gauthier, On 2-step, corank 2 nilpotent sub-Riemannian metrics. SIAM J. Control Optim, Vol. 50, No. 1, pp. 559–582, 2012

[5] M. Goresky, R. Mc Pherson, Stratified Morse Theory, Springer Verlag, 1988.

[6] M.D. Bronstein, Smoothness of roots of polynomials depending on parameters, Sibirsk Math. Zh, 20, 1979, No. 3, pp. 493,501.

[7] K. Kurdyka, L. Paunescu, Hyperbolic polynomilals and real analytic perturbation theory, Duke Mayh. Journ., 141(1),23-149, 2008.

[8] R. Montgomery, A Tour of Subriemannian Geometries, Their Geodesics and Applications (Mathematical Surveys and Monographs, Volume 91), (2002) American Mathematical Society.

[9] A. Kriegl, P.W. Michor, A. Rainer, Many Parameter Lipschitz Perturbation of Unbounded Operators, preprint.

[10] C. Romero-Melendez, J.P. Gauthier, F. Monroy-Perez, On complexity and motion planning for corank 1 surriemannian metrics, ESAIM COCV, Vol. 10, 2004, 634-655.